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Efficient Modification Scheme of Stress–Strain Tensor for Wrinkled Membranes

Kyoichi Nakashino* and M. C. Natori†
Institute of Space and Astronautical Science, Kanagawa 229-8510, Japan

A new modification scheme of the stress–strain tensor for wrinkled membranes is presented on the basis of the tension field theory. The scheme is applicable to finite element analysis of partly wrinkled membranes with arbitrary shapes. Derivation of the modification scheme rests on an introduction of the so-called "wrinkle strain" and a simplification of the virtual work equation of wrinkled membranes. Because all of the modifications required to account for wrinkling are totally confined within the stress–strain relations of membranes, the scheme can be easily implemented with existing finite element codes. Furthermore, the modified stress–strain tensor automatically leads to the consistent tangent stiffness matrix, where changes in both the wrinkling direction and the amount of wrinkliness are taken into account. Three numerical examples are treated to show the accuracy and effectiveness of the proposed modification scheme.

Nomenclature

a, b	=	inner and outer radii of annular membrane (example 2)
$\boldsymbol{C}$	=	stress–strain tensor assumed between $\boldsymbol{S}$ and $\boldsymbol{E}$
$\boldsymbol{C}'_{\mathrm{I}}$	=	modified stress–strain tensor
$\boldsymbol{C}'_{\mathrm{II}}$	=	modified stress–strain tensor in incremental form
$\boldsymbol{E}$	=	Green–Lagrange strain tensor
E	=	Young's modulus
$\boldsymbol{E}_W$	=	wrinkle strain tensor
$\boldsymbol{e}_i$	=	orthonormal base vectors of Cartesian coordinate system
$\boldsymbol{F}$	=	deformation gradient tensor
$\boldsymbol{G}_\alpha, \boldsymbol{g}_\alpha$	=	covariant base vectors of convected coordinate system in undeformed and deformed configurations
$\boldsymbol{G}^\alpha, \boldsymbol{g}^\alpha$	=	contravariant base vectors of convected coordinate system in undeformed and deformed configurations
H	=	width of rectangular membrane (example 1)
h	=	width of wrinkled region of rectangular membrane (example 1)
M	=	bending moment applied to rectangular membrane (example 1)
P	=	axial load applied to rectangular membrane (example 1)
R	=	radius of wrinkled region of annular membrane (example 2)
r^α	=	convected coordinates
$\boldsymbol{S}$	=	second Piola–Kirchhoff stress tensor
T	=	twisting moment applied to annular membrane (example 2)
t	=	thickness of membrane (examples 1–3)
$\boldsymbol{t}$	=	unit vector along to wrinkling direction
$\boldsymbol{w}$	=	unit vector transverse to wrinkling direction
β	=	physical amount of wrinkliness
γ	=	parameter associated with β, defined by Eq. (28)
θ	=	parameter associated with wrinkling direction, introduced in Eq. (23)
κ	=	overall curvature of rectangular membrane (example 1)
ν	=	Poisson's ratio
$\boldsymbol{\sigma}$	=	Cauchy stress tensor
σ_0	=	uniform stress applied to membrane (examples 1 and 2)
ϕ	=	angle of twist of rigid hub (example 2)
{ }	=	second-order tensor expressed in column vector form
[ ]	=	fourth-order tensor expressed in matrix form

Superscripts

$\dot{}$	=	material time derivative
$'$	=	variables modified to account for wrinkling

I. Introduction

RECENTLY, thin membranes attract growing interests in the field of space structure engineering because of their abilities such as high packaging ratio and lightweight property. Unlike other semirigid structural components, such as plates or shells, membranes have little resistance against compression and thus are easy to wrinkle. Because wrinkling significantly affects both static and dynamic characteristics of membrane structures, modeling of wrinkled membranes has been a subject of interest for many years.

A suitable way to treat wrinkled membranes is provided by the tension-field (TF) theory.[1] In the TF theory, a thin membrane is idealized as a membrane with zero bending stiffness. Theoretically, such an idealized membrane cannot sustain any compressive stresses. When compressive stresses are about to be introduced in the membrane, they are immediately released by out-of-plane deformations, that is, the membrane wrinkles. Wrinkled regions of the membrane, therefore, are supposed to be in uniaxial tension state, where stresses perpendicular to wrinkle lines equal zero. As just mentioned, the TF theory postulates that membranes have no bending stiffness. As a result, it cannot predict the details of wrinkles themselves (i.e., the amplitude and wavelength of the wrinkles). Even with this limitation, studies show that the TF theory provides accurate predictions for partly wrinkled membranes.[2,3]

Since the early 1970s, many studies have been conducted on the finite element analysis of partly wrinkled membranes in the light of the TF theory. In most of these studies, wrinkling phenomenon is simulated by modifying the stress–strain relations of membranes. Miller et al.[4] presented a modified elasticity matrix that produces a uniaxial stress field and performed finite element calculations of partly wrinkled membranes applying the modified

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*Research Fellow, Space Structures and Materials, Japan Aerospace Exploration Agency, 3-1-1 Yoshinodai, Sagamihara; nakasino@taurus.eng.isas.jaxa.jp. Member AIAA.

†Professor, Space Structures and Materials, Japan Aerospace Exploration Agency, 3-1-1 Yoshinodai, Sagamihara. Associate Fellow AIAA.

matrix to wrinkled regions. Liu et al.[5] proposed a penalty parameter modified material model, in which the stress–strain tensor of wrinkled membranes is modified via a penalty parameter. Similar approaches to Miller's or Liu's are adopted in Refs. 6–8. Although these modification schemes are easily integrated into finite element codes, the physical meaning of the modified stress–strain relations is somewhat obscure because the stress–strain relations of membranes do not change when wrinkling occurs. Another shortcoming of these modification schemes is that the nonlinearities originated from wrinkling are not incorporated into the tangent stiffness matrix. Hence, the convergence properties of solution processes can be deteriorated. Roddeman et al.[9] presented an alternative model of wrinkled membranes, in which the deformation gradient tensor is modified in order to account for the TF responses of membranes. Since then, several researchers have applied Roddeman model to the finite element analyses of partly wrinkled membranes.[10–13] In these studies, efforts were made to calculate the tangent stiffness matrix that is consistent with Roddeman model. However, owing to the complexity of the formulation involved the calculation of the consistent tangent stiffness matrix is not straightforward. In Ref. 11, Roddeman utilized numerical differentiation in order to obtain an approximated value of the tangent stiffness matrix. The same approach was also adopted by Muttin.[12] Jeong and Kwak[13] circumvented the direct calculation of the consistent tangent stiffness matrix by making use of linear complementarity problem formulations. Recently, Lu et al.[14] have presented explicit formulas that precisely evaluate the consistent tangent stiffness matrix. In their formulas, the matrix is calculated entry by entry through a series of tensor operations. Schoop et al.[15] have proposed another calculation algorithm. Though Schoop's formulations are well organized, the calculation still involves rather complicated procedures. Other numerical approaches for partly wrinkled membranes have been reported in the literature. In Ref. 16, energy relaxation[17] is utilized to model the TF responses of membranes, and analysis of pneumatic membrane structures is successfully presented. Recently, Ding et al.[18,19] have presented a new membrane model with modified elasticity matrix, whereby taut, wrinkled, and slackened states of membranes are automatically characterized. Their approach is based on an optimization scheme, and thus convergence problems encountered in the conventional iterative process are avoided.

In the present paper, it is shown that the modification of the deformation gradient tensor in Roddeman model can be replaced by a formal modification of the stress–strain tensor. The modified stress–strain tensor automatically leads to the consistent tangent stiffness matrix, which is substantially identical to that obtained by Lu's formulas. As shown in this paper, the modification scheme of the stress–strain tensor is expressed in terms of simple matrix-vector forms, and therefore, the calculation of the consistent tangent stiffness matrix is much simpler compared to the previous approaches. To evaluate the performance of the proposed modification scheme, two-dimensional problems of partly wrinkled membranes are solved by the finite element method based on this scheme.

II. Tension Field Theory of Wrinkled Membranes

A. Wrinkling Criterion

In the analysis of those membranes, which can be partly wrinkled or slackened, we must determine whether the membrane is taut, wrinkled, or slackened at a particular point (or a Gauss integration point in case of finite element calculations). To date, several criteria have been proposed to determine the state of the membrane. In the present paper, we adopt the following mixed stress–strain criterion, which is considered the most reasonable:

$$S_{\min} > 0 \Rightarrow \text{taut} \tag{1a}$$

$$E_{\max} > 0, \qquad S_{\min} \le 0 \Rightarrow \text{wrinkled} \tag{1b}$$

$$E_{\max} \le 0 \Rightarrow \text{slackened} \tag{1c}$$

where $S_{\min}$ denotes the minimum principal value of the second Piola–Kirchhoff stress and $E_{\max}$ the maximum principal value of the Green–Lagrange strain. Detailed explanations of the preceding criteria are found in Refs. 5 and 20.

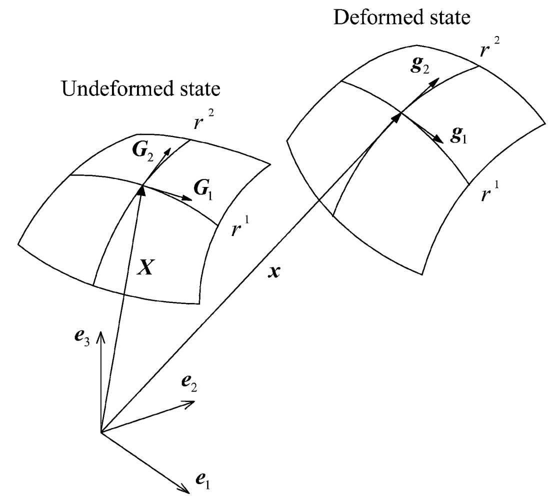

Fig. 1 Deformation of membrane.

B. Membrane Kinematics

In this subsection, we establish the basic equations of membrane kinematics, neglecting the wrinkling effects. Derivation using the tensorial components in the convected coordinate system[21] is presented. The equations presented here are directly applicable to taut regions of membranes because wrinkles are absent in these regions.

Consider a deformation of a membrane placed in three-dimensional Cartesian coordinate system $\boldsymbol{e}_i$ $(i = 1, 2, 3)$, as shown in Fig. 1. The convected coordinate system r^α $(\alpha = 1, 2)$ is employed to locate a point on the membrane midsurface. Here and in what follows, we suppose that Greek indices run through values 1 and 2. The covariant base vectors in the undeformed and deformed configurations of the membrane are defined, respectively, as

$$\boldsymbol{G}_\alpha = \frac{\partial \boldsymbol{X}(r^\alpha)}{\partial r^\alpha} \tag{2}$$

$$\boldsymbol{g}_\alpha = \frac{\partial \boldsymbol{x}(r^\alpha)}{\partial r^\alpha} \tag{3}$$

where $\boldsymbol{X}(r^\alpha)$ and $\boldsymbol{x}(r^\alpha)$ are the position vectors of a point on the membrane midsurface in the undeformed and deformed configurations, respectively. The contravariant base vectors $\boldsymbol{G}^\alpha$ and $\boldsymbol{g}^\alpha$ can be determined so that the following relations are satisfied:

$$\boldsymbol{G}^\alpha \cdot \boldsymbol{G}_\beta = \delta^\alpha_\beta \tag{4}$$

$$\boldsymbol{g}^\alpha \cdot \boldsymbol{g}_\beta = \delta^\alpha_\beta \tag{5}$$

where δ^α_β denotes the Kronecker delta ($\delta^\alpha_\beta = 1$ for $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ for $\alpha \ne \beta$). The deformation gradient tensor is given by

$$\boldsymbol{F} = \boldsymbol{g}_\alpha \otimes \boldsymbol{G}^\alpha \tag{6}$$

The Green–Lagrange strain of the membrane is then determined by

$$\boldsymbol{E} = E_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta = \tfrac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) \tag{7}$$

The second Piola–Kirchhoff stress $\boldsymbol{S}$ can be obtained from the following stress–strain relaions:

$$\boldsymbol{S} = \boldsymbol{C} : \boldsymbol{E} \qquad \text{or} \qquad S^{\alpha\beta} = C^{\alpha\beta\xi\eta}E_{\xi\eta} \tag{8}$$

In Eq. (8), $S^{\alpha\beta}$ and $C^{\alpha\beta\xi\eta}$ represent the contravariant components of $\boldsymbol{S}$ and $\boldsymbol{C}$, respectively, that is,

$$\boldsymbol{S} = S^{\alpha\beta}\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta \tag{9}$$

$$\boldsymbol{C} = C^{\alpha\beta\xi\eta}\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta \otimes \boldsymbol{G}_\xi \otimes \boldsymbol{G}_\eta \tag{10}$$

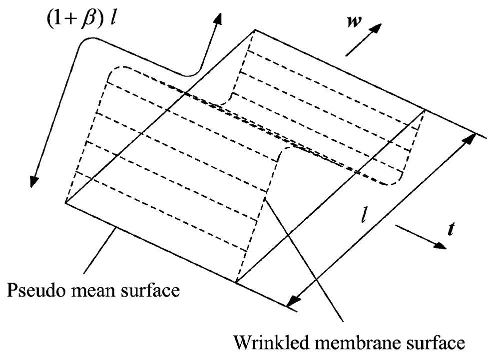

Fig. 2 Wrinkled membrane.

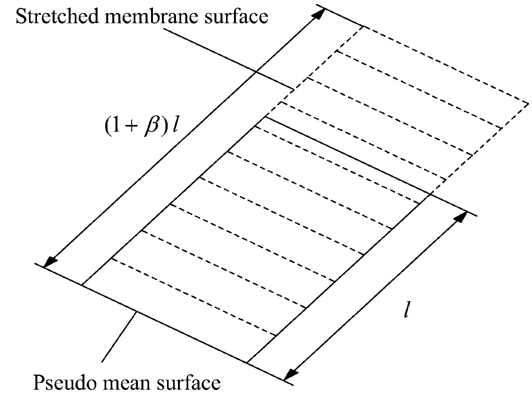

Fig. 3 Stretching of wrinkled membrane.

The calculation of the contravariant components $C^{\alpha\beta\xi\eta}$ from a usual elasticity matrix is described in Appendix. The Cauchy stress $\boldsymbol{\sigma}$ of the membrane is given, by definition, as

$$\boldsymbol{\sigma} = (1/\det \boldsymbol{F})\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T \tag{11}$$

Strains and stresses in taut regions of the membrane can be evaluated by Eqs. (7), (8), and (11). In wrinkled regions, however, these equations need to be modified in order to account for the wrinkling effects.

C. Roddeman Model

Roddeman et al.[9,10] have proposed a wrinkling model in which the deformation gradient tensor in wrinkled regions is modified so that the stress field in these regions is consistent with the TF theory. This subsection outlines the wrinkling model proposed by Roddeman.

Consider a small part of a wrinkled membrane shown in Fig. 2. In the figure, a wrinkled membrane is represented by a surface with dotted lines. A pseudo mean surface, resulting from removing the wrinkles, is also indicated by a rectangular plane with solid lines. Let $\boldsymbol{t}$ and $\boldsymbol{w}$ denote unit vectors along and transverse to the wrinkles. From a basic assumption in the TF theory, the vectors $\boldsymbol{t}$ and $\boldsymbol{w}$ coincide with the principal axes of the Cauchy stress of the membrane. It is also supposed that the Cauchy stress in the $\boldsymbol{t}$ direction is positive and that in the $\boldsymbol{w}$ direction equals zero. Namely, the membrane is in uniaxial tension state. The fine shape of the wrinkled membrane, formally drawn in the figure, cannot be uniquely determined within the framework of the TF theory because the TF theory postulates that membranes have no bending stiffness. Alternatively, we attempt to approximate the deformation of the wrinkled membrane by the pseudo mean surface. Consequently, the deformation gradient $\boldsymbol{F}$ to be considered becomes that corresponding to the pseudo mean surface. However, the Cauchy stress $\boldsymbol{\sigma}$ calculated from this deformation gradient does not satisfy the uniaxial tension conditions just stated. This is because the deformation gradient $\boldsymbol{F}$ ignores the out-of-plane deformation of the wrinkled membrane and thus underestimates the length $(1+\beta)l$ of the wrinkled membrane in the $\boldsymbol{w}$ direction (Fig. 2). To recover the uniaxial tension conditions, the deformation gradient $\boldsymbol{F}$ has to be modified according to

$$\boldsymbol{F}' = (\boldsymbol{I} + \beta \boldsymbol{w} \otimes \boldsymbol{w}) \cdot \boldsymbol{F} \tag{12}$$

where $\boldsymbol{I}$ denotes the identity tensor and β represents a measure of the amount of wrinkliness. A physical interpretation of the term $(\boldsymbol{I} + \beta \boldsymbol{w} \otimes \boldsymbol{w})$ is that it stretches the wrinkled membrane surface along to $\boldsymbol{w}$ until its wrinkles just vanish, as shown in Fig. 3. This stretching involves rigid-body movements only because membranes are supposed to possess no bending stiffness in the TF theory. Consequently, strains and stresses in the stretched membrane remain the same as those in the wrinkled membrane.

Using the modified deformation gradient $\boldsymbol{F}'$, the Green–Lagrange strain $\boldsymbol{E}'$ of the stretched membrane, equivalent to that of the wrinkled membrane, can be obtained as

$$\boldsymbol{E}' = E'_{\alpha\beta}\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta = \tfrac{1}{2}(\boldsymbol{F}'^T \cdot \boldsymbol{F}' - \boldsymbol{I}) = \boldsymbol{E} + \boldsymbol{E}_W \tag{13}$$

where

$$\boldsymbol{E}_W = \tfrac{1}{2}\beta(\beta + 2)\hat{\boldsymbol{w}} \otimes \hat{\boldsymbol{w}} \tag{14}$$

$$\hat{\boldsymbol{w}} = \boldsymbol{w} \cdot \boldsymbol{F} = \left(w_\alpha \boldsymbol{g}^\alpha\right) \cdot \boldsymbol{F} = w_\alpha \boldsymbol{G}^\alpha \tag{15}$$

From Eqs. (7) and (13–15), the covariant components of $\boldsymbol{E}'$ can be written as

$$E'_{\alpha\beta} = E_{\alpha\beta} + \tfrac{1}{2}\beta(\beta + 2)w_\alpha w_\beta \tag{16}$$

In Eq. (13), $\boldsymbol{E}_W$ represents an additional strain term introduced as a consequence of accounting for the wrinkling effects. This strain term is sometimes referred to as the wrinkle strain. The wrinkle strain $\boldsymbol{E}_W$ results from the imaginary rigid-body movements to stretch the wrinkled membrane. The modified second Piola–Kirchhoff stress $\boldsymbol{S}'$ is obtained from Eq. (13) as

$$\boldsymbol{S}' = \boldsymbol{C} : \boldsymbol{E}' \qquad \text{or} \qquad S'^{\alpha\beta} = C^{\alpha\beta\xi\eta}E'_{\xi\eta} \tag{17}$$

where

$$\boldsymbol{S}' = S'^{\alpha\beta}\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\beta \tag{18}$$

Hence, the modified Cauchy stress $\boldsymbol{\sigma}'$ is given as

$$\boldsymbol{\sigma}' = (1/\det \boldsymbol{F}')\boldsymbol{F}' \cdot \boldsymbol{S}' \cdot \boldsymbol{F}'^T \tag{19}$$

Because the wrinkled membrane is supposed to be in uniaxial tension state, the following condition must hold:

$$\boldsymbol{\sigma}' \cdot \boldsymbol{w} = 0 \tag{20}$$

The variables β and $\boldsymbol{w}$, introduced in Eq. (12), are determined so that the preceding uniaxial tension condition is satisfied.

D. Calculation of Strains and Stresses of Wrinkled Membranes

To calculate $\boldsymbol{E}'$ and $\boldsymbol{S}'$, we must obtain β and $\boldsymbol{w}$ that satisfy the uniaxial tension condition (20). This condition, along with Eq. (19), consists of coupled nonlinear equations with variables β and $\boldsymbol{w}$. However, these equations can be uncoupled under the condition that the membrane material is linear elastic.[14]

Substituting Eq. (19) into Eq. (20) and using Eqs. (12) and (15), we obtain

$$\boldsymbol{S}' \cdot \hat{\boldsymbol{w}} = 0 \tag{21}$$

Introducing new variables θ and α, we can write the covariant components of $\boldsymbol{w}$ as

$$w_1 = \alpha n_1, \qquad w_2 = \alpha n_2 \qquad \alpha \neq 0 \tag{22}$$

$$n_1 = \cos\theta, \qquad n_2 = \sin\theta \tag{23}$$

Using Eqs. (15), (18), (22), and (23), the relation (21) proves to be equivalent to

$$S'^{\alpha\beta} n_\alpha n_\beta = 0 \tag{24}$$

$$S'^{\alpha\beta} m_\alpha n_\beta = 0 \tag{25}$$

where

$$m_1 = -\sin\theta, \qquad m_2 = \cos\theta \tag{26}$$

The substitution of Eq. (22) into Eq. (16) yields

$$E'_{\alpha\beta} = E_{\alpha\beta} + \gamma n_\alpha n_\beta \tag{27}$$

$$\gamma = \tfrac{1}{2}\alpha^2 \beta(\beta + 1) \tag{28}$$

From Eqs. (27) and (17), we obtain

$$S'^{\alpha\beta} = C^{\alpha\beta\xi\eta}(E_{\xi\eta} + \gamma n_\xi n_\eta) = S^{\alpha\beta} + \gamma C^{\alpha\beta\xi\eta} n_\xi n_\eta \tag{29}$$

Hence, the relations (24) and (25) become

$$S^{\alpha\beta} n_\alpha n_\beta + \gamma C^{\alpha\beta\xi\eta} n_\alpha n_\beta n_\xi n_\eta = 0 \tag{30}$$

$$S^{\alpha\beta} m_\alpha n_\beta + \gamma C^{\alpha\beta\xi\eta} m_\alpha n_\beta n_\xi n_\eta = 0 \tag{31}$$

For linear elastic materials, the components $C^{\alpha\beta\xi\eta}$ are constant, and Eq. (30) is easily solved for γ to yield

$$\gamma = -\frac{S^{\xi\eta} n_\xi n_\eta}{C^{\sigma\zeta\tau\nu} n_\sigma n_\zeta n_\tau n_\nu} \tag{32}$$

From Eqs. (31) and (32), we obtain the following equation:

$$f(\theta) \equiv S^{\alpha\beta} m_\alpha n_\beta - \frac{S^{\xi\eta} n_\xi n_\eta}{C^{\sigma\zeta\tau\nu} n_\sigma n_\zeta n_\tau n_\nu} C^{\alpha\beta\lambda\pi} m_\alpha n_\beta n_\lambda n_\pi = 0 \tag{33}$$

The preceding equation, derived from the uniaxial tension condition (20), contains only one unknown variable θ and is easily solved by an appropriate numerical method. An interval of θ within which the solution exists can be limited beforehand by utilizing the conditions $\beta > 0$ and $\boldsymbol{t} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{t} > 0$ (Ref. 14). Equation (33) is to be utilized later on in the calculation of the consistent tangent stiffness matrix. Once the solution θ is obtained, the modified Green–Lagrange strain $\boldsymbol{E}'$ can be calculated from Eqs. (23), (27), and (32). The modified second Piola–Kirchhoff stress $\boldsymbol{S}'$ follows from Eq. (17).

E. Virtual Work Equation of Wrinkled Membrane

Generally, the nonlinear finite element formulation is established by the use of the virtual work equation. The tangent stiffness matrix and the internal force vector required for performing Newton–Raphson iteration are derived from linearization and discretization of the virtual work equation. In this subsection, we examine the virtual work equation of wrinkled membranes in the total Lagrangian framework and derive a basic equation for the calculation of the consistent tangent stiffness matrix.

When the membrane is in taut state, the virtual work equation takes its usual form. In the context of the total Lagrangian formulation, the equation is expressed as

$$\int_V (\boldsymbol{S} : \delta\boldsymbol{E})\,\mathrm{d}V = \delta R \tag{34}$$

where V denotes the volume of the undeformed membrane, δR denotes the external virtual work, and δ signifies the variation caused by the virtual displacement. To perform Newton–Raphson iteration process, the left-hand side of Eq. (34) must be linearized. The linearized form is expressed as

$$\int_V (\boldsymbol{S} : \delta\boldsymbol{E})\,\mathrm{d}V \to \int_V (\dot{\boldsymbol{S}} : \delta\boldsymbol{E})\,\mathrm{d}V + \int_V \boldsymbol{S} : (\delta\boldsymbol{E})^{\cdot}\,\mathrm{d}V \tag{35}$$

where the superscript dots denote the material time derivative. Substituting the stress–strain relation (8) together with its incremental form

$$\dot{\boldsymbol{S}} = \boldsymbol{C} : \dot{\boldsymbol{E}} \tag{36}$$

into Eq. (35), we obtain

$$\int_V (\boldsymbol{S} : \delta\boldsymbol{E})\,\mathrm{d}V \to \int_V (\boldsymbol{C} : \dot{\boldsymbol{E}}) : \delta\boldsymbol{E}\,\mathrm{d}V + \int_V (\boldsymbol{C} : \boldsymbol{E}) : (\delta\boldsymbol{E})^{\cdot}\,\mathrm{d}V \tag{37}$$

The tangent stiffness matrix for taut regions of the membrane can be obtained by discretizing Eq. (37).

When the membrane is in wrinkled state, the virtual work equation (34) is replaced by

$$\int_V \boldsymbol{S}' : \delta\boldsymbol{E}'\mathrm{d}V = \int_V \boldsymbol{S}' : (\delta\boldsymbol{E} + \delta\boldsymbol{E}_W)\,\mathrm{d}V = \delta R \tag{38}$$

Equation (38), if directly discretized by finite elements, yields quite complicated expressions.[14] However, this equation can be simplified in the following manner. Equation (21) means that $\hat{\boldsymbol{w}}$ is the principal direction of $\boldsymbol{S}'$ and the principal value corresponding to $\hat{\boldsymbol{w}}$ equals zero. Therefore, the modified second Piola–Kirchhoff stress $\boldsymbol{S}'$ can be decomposed into

$$\boldsymbol{S}' = 0 \cdot \hat{\boldsymbol{w}} \otimes \hat{\boldsymbol{w}} + S'_{\mathrm{I}} \boldsymbol{v} \otimes \boldsymbol{v} = S'_{\mathrm{I}} \boldsymbol{v} \otimes \boldsymbol{v} \tag{39}$$

where $\boldsymbol{v}$ is the other principal direction perpendicular to $\hat{\boldsymbol{w}}$ and S'_{I} is the principal value corresponding to $\boldsymbol{v}$. Meanwhile, the virtual wrinkle strain $\delta\boldsymbol{E}_W$ is obtained from Eq. (14) as

$$\delta\boldsymbol{E}_W = \tfrac{1}{2}\beta(\beta+2)(\delta\hat{\boldsymbol{w}} \otimes \hat{\boldsymbol{w}} + \hat{\boldsymbol{w}} \otimes \delta\hat{\boldsymbol{w}}) + (\beta+1)\delta\beta(\hat{\boldsymbol{w}} \otimes \hat{\boldsymbol{w}}) \tag{40}$$

Substituting Eqs. (39) and (40) into Eq. (38) and noting that

$$\hat{\boldsymbol{w}} \cdot \boldsymbol{v} = 0$$

we obtain a simpler form of the virtual work equation:

$$\int_V \boldsymbol{S}' : (\delta\boldsymbol{E} + \delta\boldsymbol{E}_W)\,\mathrm{d}V = \int_V \boldsymbol{S}' : \delta\boldsymbol{E}\,\mathrm{d}V = \delta R \tag{41}$$

Equation (41) means that the virtual wrinkle strain $\delta\boldsymbol{E}_W$ has no contribution to the internal virtual work. This is explained by the fact that the wrinkle strain corresponds to the rigid-body movements to stretch the wrinkled membrane and thus does not alter the strain energy of the membrane. Obviously, the decomposed forms of $\boldsymbol{E}_W$ and $\boldsymbol{S}'$, that is, Eqs. (14) and (39), play the key role in the preceding simplification. The derivation of Eqs. (14) and (39) is credited to Lu et al.,[14] though they did not utilize these equations to simplify the internal virtual work. To obtain the tangent stiffness matrix, we linearize the left-hand side of Eq. (41) and obtain the following form:

$$\begin{aligned}\int_V \boldsymbol{S}' : \delta\boldsymbol{E}\,\mathrm{d}V &\to \int_V (\dot{\boldsymbol{S}}' : \delta\boldsymbol{E})\,\mathrm{d}V + \int_V \boldsymbol{S}' : (\delta\boldsymbol{E})^{\cdot}\,\mathrm{d}V \\ &= \int_V (\boldsymbol{C} : \dot{\boldsymbol{E}}') : \delta\boldsymbol{E}\,\mathrm{d}V + \int_V (\boldsymbol{C} : \boldsymbol{E}') : (\delta\boldsymbol{E})^{\cdot}\,\mathrm{d}V\end{aligned} \tag{42}$$

III. Proposed Modification Scheme of Stress–Strain Tensor

A. Modified Stress–Strain Tensor

In the preceding section, we derive Eq. (42) as a linearized form of the virtual work equation of wrinkled membranes. Discretizing Eq. (42) yields the tangent stiffness matrix for wrinkled regions of the membrane, but this can result in cumbersome calculations because discretization of $\boldsymbol{E}'$ and $\dot{\boldsymbol{E}}'$ cannot be carried out in a straightforward manner. However, it is proved that the tangent stiffness matrix can be effectively calculated by formally modifying the usual stress–strain tensor $\boldsymbol{C}$, in which case direct discretization of $\boldsymbol{E}'$ and

$\dot{\boldsymbol{E}}'$ is not required. In this section, we establish formulas for the modification of $\boldsymbol{C}$ just stated.

The stress–strain relations of wrinkled membranes are expressed as

$$\boldsymbol{S}' = \boldsymbol{C} : \boldsymbol{E}' \tag{43}$$

$$\dot{\boldsymbol{S}}' = \boldsymbol{C} : \dot{\boldsymbol{E}}' \tag{44}$$

Now, suppose we obtain fourth-order tensors $\boldsymbol{C}'_{\mathrm{I}}$ and $\boldsymbol{C}'_{\mathrm{II}}$ that formally satisfy the following relations:

$$\boldsymbol{S}' = \boldsymbol{C}'_{\mathrm{I}} : \boldsymbol{E} \tag{45}$$

$$\dot{\boldsymbol{S}}' = \boldsymbol{C}'_{\mathrm{II}} : \dot{\boldsymbol{E}} \tag{46}$$

In this case, Eq. (42) can be rewritten as

$$\int_V (\boldsymbol{C}'_{\mathrm{II}} : \dot{\boldsymbol{E}}) : \delta\boldsymbol{E}\,\mathrm{d}V + \int_V (\boldsymbol{C}'_{\mathrm{I}} : \boldsymbol{E}) : (\delta\boldsymbol{E})^{\cdot}\,\mathrm{d}V \tag{47}$$

Note here that the quantities related to $\boldsymbol{E}$ appearing in the preceding equation can be obtained through usual procedures not accounting for the wrinkling effects. Hence, the tangent stiffness matrix of wrinkled membranes can be obtained from Eq. (37) via the modifications

$$\boldsymbol{C}\,(\text{in the first term}) \rightarrow \boldsymbol{C}'_{\mathrm{II}}$$

$$\boldsymbol{C}\,(\text{in the second term}) \rightarrow \boldsymbol{C}'_{\mathrm{I}}$$

In the remaining part of this section, we will derive the explicit formulas for the modified stress–strain tensors $\boldsymbol{C}'_{\mathrm{I}}$ and $\boldsymbol{C}'_{\mathrm{II}}$. For convenience, we define the following vectors and matrices:

$$\{\boldsymbol{E}\} = [E_{11} \quad E_{22} \quad 2E_{12}]^T \tag{48a}$$

$$\{\boldsymbol{E}'\} = [E'_{11} \quad E'_{22} \quad 2E'_{12}]^T \tag{48b}$$

$$\{\boldsymbol{S}\} = [S^{11} \quad S^{22} \quad S^{12}]^T \tag{48c}$$

$$\{\boldsymbol{S}'\} = [S'^{11} \quad S'^{22} \quad S'^{12}]^T \tag{48d}$$

$$\boldsymbol{n}_1 = [n_1 n_1 \quad n_2 n_2 \quad 2n_1 n_2]^T \tag{48e}$$

$$\boldsymbol{n}_2 = [m_1 n_1 \quad m_2 n_2 \quad m_1 n_2 + m_2 n_1]^T \tag{48f}$$

$$\boldsymbol{n}_3 = [m_1 m_1 - n_1 n_1 \quad m_2 m_2 - n_2 n_2 \quad 2(m_1 m_2 - n_1 n_2)]^T \tag{48g}$$

$$\boldsymbol{n}_4 = [m_1 m_1 \quad m_2 m_2 \quad 2m_1 m_2]^T \tag{48h}$$

$$[\boldsymbol{C}] = \begin{bmatrix} C^{1111} & C^{1122} & C^{1112} \\ C^{2211} & C^{2222} & C^{2212} \\ C^{1211} & C^{1222} & C^{1212} \end{bmatrix} \tag{48i}$$

In what follows, we assume that the matrix $[\boldsymbol{C}]$ is symmetric. Using the preceding vectors and matrices, Eqs. (29), (32), and (33) are recast into the following forms:

$$\{\boldsymbol{S}'\} = [\boldsymbol{C}] \cdot (\{\boldsymbol{E}\} + \gamma \boldsymbol{n}_1) \tag{49}$$

$$\gamma = -\frac{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \{\boldsymbol{E}\}}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \tag{50}$$

$$f(\theta) = \boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \{\boldsymbol{E}\} + \gamma \boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1 = 0 \tag{51}$$

Likewise, Eqs. (24) and (25) are rewritten as

$$\boldsymbol{n}_1^T \cdot \{\boldsymbol{S}'\} = 0 \tag{52}$$

$$\boldsymbol{n}_2^T \cdot \{\boldsymbol{S}'\} = 0 \tag{53}$$

By Eqs. (48) and (52), the following relation holds:

$$\boldsymbol{n}_3^T \cdot \{\boldsymbol{S}'\} = \boldsymbol{n}_4^T \cdot \{\boldsymbol{S}'\} \tag{54}$$

B. Derivation of $\boldsymbol{C}'_{\mathrm{I}}$

Substituting Eq. (50) into Eq. (49), we obtain

$$\{\boldsymbol{S}'\} = [\boldsymbol{C}] \cdot \left(\{\boldsymbol{E}\} - \frac{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \{\boldsymbol{E}\}}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \boldsymbol{n}_1 \right) = [\boldsymbol{C}'_{\mathrm{I}}] \cdot \{\boldsymbol{E}\} \tag{55}$$

where

$$[\boldsymbol{C}'_{\mathrm{I}}] = [\boldsymbol{C}] - \left[1 \big/ \left(\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1\right)\right] [\boldsymbol{C}] \cdot \boldsymbol{n}_1 \cdot \boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \tag{56}$$

At this point, it becomes clear that $[\boldsymbol{C}'_{\mathrm{I}}]$ is the equivalent matrix form of the modified stress–strain tensor $\boldsymbol{C}'_{\mathrm{I}}$.

C. Derivation of $\boldsymbol{C}'_{\mathrm{II}}$

Because the Green–Lagrange strain $\boldsymbol{E}$ defined by Eq. (7) can be viewed as a function of the position vector $\boldsymbol{x}$, the following relation holds:

$$\{\dot{\boldsymbol{E}}\} = \frac{\partial \{\boldsymbol{E}\}}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} \equiv \boldsymbol{B} \cdot \dot{\boldsymbol{x}} \tag{57}$$

The matrix $\boldsymbol{B}$ defined in the preceding equation is used here rather symbolically; the explicit expression for this matrix is not required in the formulations that follow. Taking the time derivative of Eq. (49), we obtain

$$\begin{aligned} \{\dot{\boldsymbol{S}}'\} &= [\boldsymbol{C}] \cdot (\{\dot{\boldsymbol{E}}\} + \dot{\gamma} \boldsymbol{n}_1 + \gamma \dot{\boldsymbol{n}}_1) \\ &= [\boldsymbol{C}] \cdot \{\dot{\boldsymbol{E}}\} + [\boldsymbol{C}] \cdot \left(\boldsymbol{n}_1 \cdot \frac{\partial \gamma}{\partial \boldsymbol{x}} + \frac{\partial \gamma}{\partial \theta} \boldsymbol{n}_1 \cdot \frac{\partial \theta}{\partial \boldsymbol{x}} + 2\gamma \boldsymbol{n}_2 \cdot \frac{\partial \theta}{\partial \boldsymbol{x}} \right) \cdot \dot{\boldsymbol{x}} \end{aligned} \tag{58}$$

In deriving the preceding equation, we utilize the relation $\partial \boldsymbol{n}_1 / \partial\theta = 2\boldsymbol{n}_2$. Now, we have to obtain explicit formulas for three partial derivatives appearing in the right-hand side of Eq. (58). From Eqs. (50) and (57), the derivative $\partial\gamma/\partial\boldsymbol{x}$ is obtained as

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = -\frac{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{B}}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \tag{59}$$

Differentiating Eq. (50) with respect to θ gives

$$\frac{\partial \gamma}{\partial \theta} = -\frac{1}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \left[2\left(\boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \{\boldsymbol{E}\}\right) + 4\gamma \left(\boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1\right) \right] \tag{60}$$

Using Eqs. (49) and (53), we can simplify Eq. (60) to

$$\frac{\partial \gamma}{\partial \theta} = -2\gamma \frac{\boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \tag{61}$$

Taking the time derivative of the equation $f(\theta) = 0$ leads to

$$\frac{\partial \theta}{\partial \boldsymbol{x}} = -\frac{\partial f}{\partial \boldsymbol{x}} \bigg/ \frac{\partial f}{\partial \theta} \tag{62}$$

Differentiating Eq. (51) with respect to $\boldsymbol{x}$ and using Eqs. (57) and (59), we obtain the numerator in the right-hand side of Eq. (62) as

$$\frac{\partial f}{\partial \boldsymbol{x}} = \left(\boldsymbol{n}_2^T - \frac{\boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \boldsymbol{n}_1^T \right) \cdot [\boldsymbol{C}] \cdot \boldsymbol{B} \tag{63}$$

Similarly, differentiating Eq. (51) with respect to θ and using Eqs. (61), (49), and (54), we obtain the denominator in right-hand side of Eq. (62) as

$$\begin{aligned} \frac{\partial f}{\partial \theta} &\equiv f_{,\theta} = \boldsymbol{n}_4^T \cdot [\boldsymbol{C}] \cdot (\{\boldsymbol{E}\} + \gamma \boldsymbol{n}_1) \\ &\quad + 2\gamma \left[\boldsymbol{n}_2^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_2 - \frac{\left(\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_2\right)^2}{\boldsymbol{n}_1^T \cdot [\boldsymbol{C}] \cdot \boldsymbol{n}_1} \right] \end{aligned} \tag{64}$$

In deriving Eq. (64), we utilize the relations $\partial \boldsymbol{n}_1/\partial\theta = 2\boldsymbol{n}_2$ and $\partial \boldsymbol{n}_2/\partial\theta = \boldsymbol{n}_3$. Substituting Eqs. (63) and (64) into Eq. (62), we obtain the derivative $\partial\theta/\partial\boldsymbol{x}$ as

$$\frac{\partial\theta}{\partial\boldsymbol{x}} = \frac{1}{f_{,\theta}}\left(\frac{\boldsymbol{n}_1^T\cdot[\boldsymbol{C}]\cdot\boldsymbol{n}_2}{\boldsymbol{n}_1^T\cdot[\boldsymbol{C}]\cdot\boldsymbol{n}_1}\boldsymbol{n}_1^T - \boldsymbol{n}_2^T\right)\cdot[\boldsymbol{C}]\cdot\boldsymbol{B} \qquad (65)$$

Finally, the substitution of Eqs. (59), (61), and (65) into Eq. (58) gives rise to the following expression:

$$\{\dot{\boldsymbol{S}}'\} = [\boldsymbol{C}'_{\mathrm{II}}]\cdot\{\dot{\boldsymbol{E}}\} \qquad (66)$$

where

$$[\boldsymbol{C}'_{\mathrm{II}}] = [\boldsymbol{C}'_{\mathrm{I}}] + (2\gamma/f_{,\theta})[\boldsymbol{C}]\cdot\boldsymbol{b}\cdot\boldsymbol{b}^T\cdot[\boldsymbol{C}] \qquad (67)$$

$$\boldsymbol{b} = \boldsymbol{n}_2 - \frac{\boldsymbol{n}_1^T\cdot[\boldsymbol{C}]\cdot\boldsymbol{n}_2}{\boldsymbol{n}_1^T\cdot[\boldsymbol{C}]\cdot\boldsymbol{n}_1}\boldsymbol{n}_1 \qquad (68)$$

From Eq. (66), it is observed that the matrix $[\boldsymbol{C}'_{\mathrm{II}}]$ defined by Eq. (67) is the equivalent matrix form of the modified stress–strain tensor $\boldsymbol{C}'_{\mathrm{II}}$.

Remark 1: Using Eqs. (52) and (53), it is easily shown that the following relation holds:

$$\{\boldsymbol{S}'\} = [\boldsymbol{C}'_{\mathrm{II}}]\cdot\{\boldsymbol{E}\} \qquad (69)$$

Equation (69) means that the matrix $[\boldsymbol{C}'_{\mathrm{I}}]$ appearing in Eq. (55) can be replaced by $[\boldsymbol{C}'_{\mathrm{II}}]$. Hence, the tangent stiffness matrix of wrinkled membrane can be obtained from Eq. (37) by simply modifying $\boldsymbol{C}$ to $\boldsymbol{C}'_{\mathrm{II}}$. This feature can be useful for practical calculations.

Remark 2: In finite element calculations, the internal force vector is also required in addition to the tangent stiffness matrix. The internal force vector for wrinkle regions of the membrane can be obtained by discretizing the following term:

$$\int_V \boldsymbol{S}' : \delta\boldsymbol{E}\,\mathrm{d}V\left(=\int_V(\boldsymbol{C}'_{\mathrm{I}}:\boldsymbol{E}):\delta\boldsymbol{E}\,\mathrm{d}V\right)$$

Hence, the internal force vector can be obtained by modifying the usual stress–strain tensor $\boldsymbol{C}$ to $\boldsymbol{C}'_{\mathrm{I}}$ (or alternatively to $\boldsymbol{C}'_{\mathrm{II}}$).

Remark 3: From Eqs. (56) and (67), it is observed that the matrices $[\boldsymbol{C}'_{\mathrm{I}}]$ and $[\boldsymbol{C}'_{\mathrm{II}}]$ are symmetric as long as the usual stress–strain matrix $[\boldsymbol{C}]$ is symmetric.

Remark 4: In the derivations just described, we consider derivatives such as $\partial\gamma/\partial\boldsymbol{x}$ and $\partial\theta/\partial\boldsymbol{x}$. This means that changes in the wrinkling direction and the amount of wrinkliness are both taken into account in the formulations. These changes are also incorporated into the tangent stiffness matrix calculated from the modified stress–strain tensors $[\boldsymbol{C}'_{\mathrm{I}}]$ and $[\boldsymbol{C}'_{\mathrm{II}}]$.

Remark 5: $[\boldsymbol{C}'_{\mathrm{I}}]$ and $[\boldsymbol{C}'_{\mathrm{II}}]$ are the matrices containing contravariant components of the modified stress–strain tensors $\boldsymbol{C}'_{\mathrm{I}}$ and $\boldsymbol{C}'_{\mathrm{II}}$. Cartesian components of these tensors, if required, can be obtained by transformation of coordinates.

IV. Finite Element Analysis

A. Treatment of Wrinkling

In the finite element calculations, the tangent stiffness matrix and the internal force vector for each element are usually obtained by Gauss numerical integration, where stresses, strains, and other related quantities are evaluated at each integration point. In case that membranes are allowed to be partly wrinkled or slackened, we must properly evaluate the preceding quantities, at each integration point, in accordance with the state of the membrane (i.e., taut, wrinkled, or slackened). In the context of the proposed modification scheme, this can be accomplished by modifying the usual stress–strain tensor $[\boldsymbol{C}]$ according to the following.

Taut:

$$[\boldsymbol{C}] \Rightarrow [\boldsymbol{C}] \qquad (70a)$$

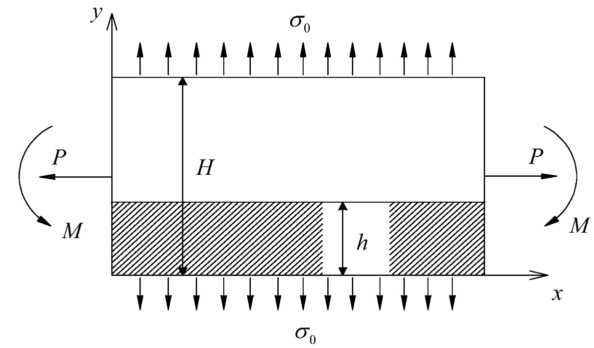

Fig. 4 Rectangular membrane subjected to in-plane bending moment.

Wrinkled:

$$[\boldsymbol{C}] \Rightarrow [\boldsymbol{C}'_{\mathrm{II}}] \qquad (70b)$$

Slackened:

$$[\boldsymbol{C}] \Rightarrow [O] \qquad (70c)$$

where $[O]$ denotes a 3×3 zero matrix. The state of the membrane can be determined at each integration point using the mixed stress–strain criterion (1). Equations (70) establish the treatment of wrinkling used in this study.

B. Example 1: In-Plane Bending of a Rectangular Membrane

Consider in-plane bending of a rectangular membrane with width H and thickness t, as shown in Fig. 4. The membrane is subjected to uniform stress σ_0 in the y direction, axial load $P = \sigma_0 t H$ in the x direction, and bending moment M on its two ends. With increasing moment M, a band of vertical wrinkles of width h grows from the lower edge of the membrane. This simple problem has served as a benchmark for various numerical approaches to the analysis of partly wrinkled membranes.[4,13,14,19] An analytical solution for this problem is also available, which is found in Ref. 22. As stated in the reference, the bandwidth of the wrinkled region is determined by

$$\frac{h}{H} = \begin{cases} 0; & M/PH < 1/6 \\ 3M/PH - 1/2; & 1/6 \le M/PH < 1/2 \end{cases} \qquad (71)$$

The normal stress σ_x in the x direction of the membrane is expressed as

$$\frac{\sigma_x}{\sigma_0} = \begin{cases} \dfrac{2(y/H - h/H)}{(1-h/H)^2}; & h/H < y/H \le 1 \\ 0; & 0 \le y/H \le h/H \end{cases} \qquad (72)$$

If κ denotes the overall curvature of the membrane acting as a beam, the moment-curvature relation is given by

$$\frac{2M}{PH} = \begin{cases} EH^2t\kappa/6P; & EH^2t\kappa/2P \le 1 \\ 1 - \dfrac{2}{3}\sqrt{\dfrac{2P}{EH^2t\kappa}}; & EH^2t\kappa/2P > 1 \end{cases} \qquad (73)$$

A finite element model for this problem is shown in Fig. 5. The model consists of 55 isoparametric nine-node membrane elements. By symmetry, only the right half of the membrane is modeled. The moment M and the axial load P are replaced by two equivalent concentrated loads. The treatment of wrinkling, stated at the beginning of this section, is applied to all of the membrane elements except those located in the shaded area. We assume that these five elements can resist compressive stresses and are never wrinkled nor slackened. Without this exception, the five elements tend to be slackened, thus failing to transverse the bending moment properly.

A comparison of the overall moment-curvature relation of the membrane between analytical and numerical results is shown in Fig. 6. The curvatures corresponding to numerical results are obtained by assuming that the nodal displacements in the y direction

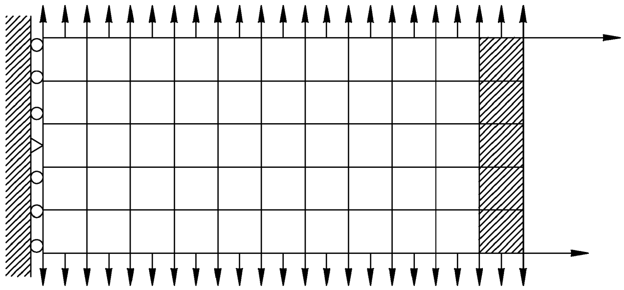

Fig. 5 Finite element model of a rectangular membrane.

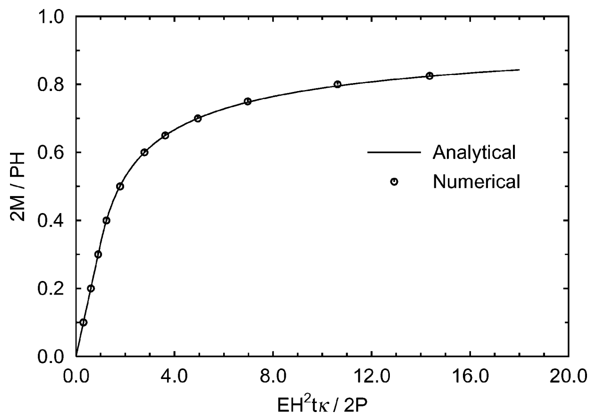

Fig. 6 Moment-curvature relation for in-plane bending of a rectangular membrane.

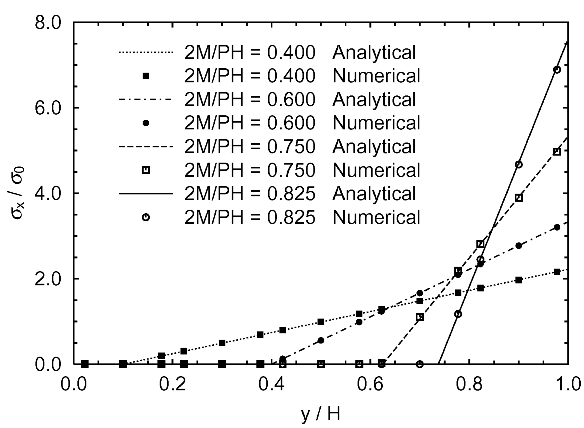

Fig. 7 Normal stress σ_x vs vertical position y for in-plane bending of a rectangular membrane.

are described by a quadratic function of x. Shown in Fig. 7 is a comparison of analytical and numerical results for σ_x vs y for four different values of M. As shown in Figs. 6 and 7, analytical and numerical results are in good agreement with each other.

Convergence rates of the iterative solution process are illustrated in Fig. 8, where solution errors are plotted vs the number of Newton–Raphson iterations. The solution errors are calculated as the norm of the residual force vector normalized by the norm of the external force vector. The convergence tolerance of 10^{-12} is adopted in the calculations. Despite the strict convergence tolerance, the iterations converge with relatively few steps, owing to the consistency of the tangent stiffness matrix.

C. Example 2: In-Plane Torsion of an Annular Membrane

An annular membrane of thickness t is attached to a rigid hub at the inner edge and to a guard ring at the outer edge, as shown in Fig. 9. The membrane is pretensioned by a uniform stress σ_0. The rigid hub is rotated by a moment T through an angle ϕ. As the moment T increases, wrinkles begin to form around the hub out to

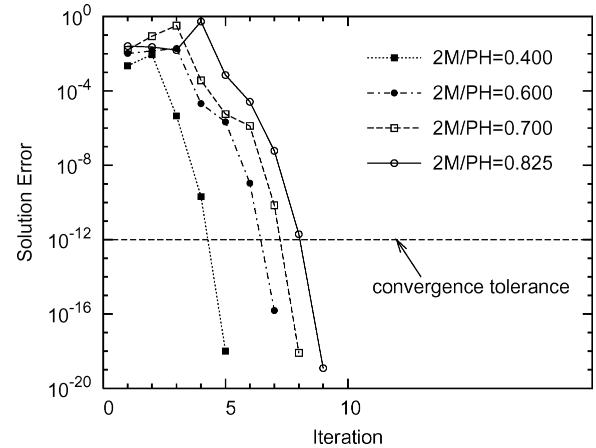

Fig. 8 Convergence behavior for in-plane bending of a rectangular membrane.

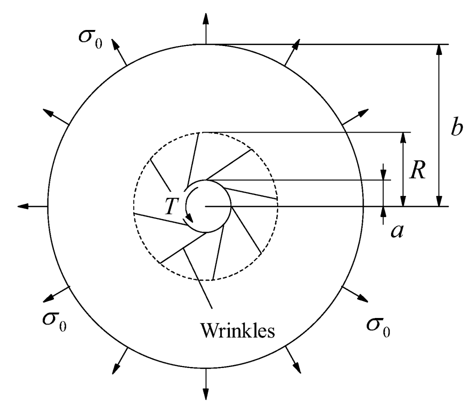

Fig. 9 Annular membrane subjected to twisting moment through an attached hub.

some radius R. An analytical solution for this problem is presented in Ref. 3. It is shown in the reference that the relation between the wrinkle radius R and the prescribed moment T is governed by

$$1/A + 1/B - \ln(B/A) - 2/3 = 0 \tag{74}$$

$$\bar{C}_1^2 - \left[1 + 2\bar{C}_1(a/b)^2\right]^2 \bar{R}^4 + \bar{T}^2 = 0 \tag{75}$$

$$\bar{C}_2 = \left\{\bar{R} + [1/\bar{R} + 2\bar{R}(a/b)^2]\bar{C}_1\right\}^2 + (\bar{T}/\bar{R})^2 \tag{76}$$

where

$$\bar{T} = T\big/2\pi a^2 \sigma_0 t, \qquad \bar{R} = R/a$$

$$A = \bar{C}_2\big/\bar{T}^2 - 1, \qquad B = \bar{R}^2\bar{C}_2\big/\bar{T}^2 - 1$$

The preceding equations are derived on the assumption that Poisson's ratio of the membrane equals $\frac{1}{3}$. The relation between the moment T and the angle of twist ϕ of the hub is governed by

$$\bar{\phi} = \frac{3\bar{T}}{8(1 - a^2/b^2)}\left[\frac{1/\bar{R}^2 - 1}{B} + \ln\left(\frac{B}{A}\right) + \frac{1}{\bar{R}^2} - \frac{8}{3}\left(\frac{a}{b}\right)^2 + \frac{5}{3}\right] \tag{77}$$

where

$$\bar{\phi} = \frac{3E\phi}{4\sigma_0(1 - a^2/b^2)}$$

The principal stresses σ_1 and σ_2 ($\sigma_1 > \sigma_2$) of the membrane can be expressed as

$$\frac{\sigma_1}{\sigma_0} = \begin{cases} \dfrac{\bar{C}_2/\bar{r}}{\sqrt{\bar{C}_2 - \bar{T}^2/\bar{r}^2}}; & \bar{r} < \bar{R} \\ 2\bar{C}_1(a/b)^2 + 1 + \sqrt{\bar{C}_1^2 + \bar{T}^2}\Big/\bar{r}^2; & \bar{r} \geq \bar{R} \end{cases} \tag{78}$$

$$\frac{\sigma_2}{\sigma_0} = \begin{cases} 0; & \bar{r} < \bar{R} \\ 2\bar{C}_1(a/b)^2 + 1 - \sqrt{\bar{C}_1^2 + \bar{T}^2}\Big/\bar{r}^2; & \bar{r} \geq \bar{R} \end{cases} \tag{79}$$

where

$$\bar{r} = r/a$$

and r is the radial distance from the center of the hub.

A finite element model for this problem is shown in Fig. 10. The model is discretized by 324 nine-node membrane elements. Material property of the membrane is assumed to be linear isotropic with Poisson's ratio of $\nu = \frac{1}{3}$. A finite element analysis of this problem is carried out in the following manner. First, a uniform stress σ_0 is prescribed and then replaced by equivalent nodal displacements imposed on the outer edge. Second, a certain value of the wrinkle radius R is given. The twisting angle ϕ corresponding to the given R is calculated from Eqs. (74–77). The calculated angle ϕ is applied to the finite element model as nodal displacements along the inner edge. Following the procedure just described, numerical analyses with various values of R are performed.

The numerical results obtained by the present treatment of wrinkling are shown in Figs. 11–13 together with the corresponding analytical results. Figures 11 and 12 show comparisons of numerical and analytical results for the principal stress distributions of the membrane. Shown in Fig. 13 is a comparison of analytical and numerical values for the moment $\bar{T}$ as a function of the twisting angle $\bar{\phi}$. The numerical values of the moment $\bar{T}$ are computed from shear components of the numerical stresses. Figures 11–13 show good agreement between numerical and analytical results. Regarding the solution processes, good convergence properties are observed as in the preceding example. Figure 14 shows the convergence behavior observed in the calculations. The total number of iterations required for convergence is less than or equal to five in all cases.

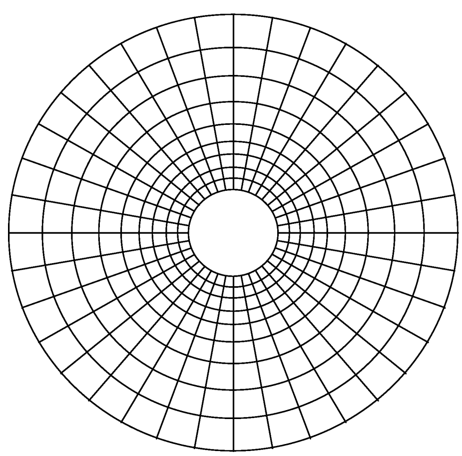

Fig. 10 Finite element model of an annular membrane.

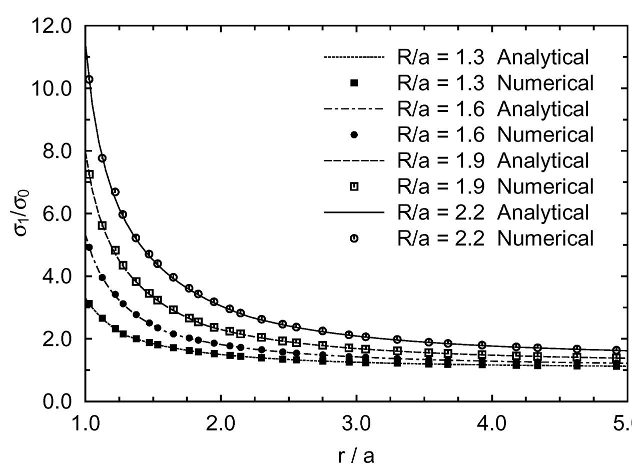

Fig. 11 Maximum principal stress σ_1 vs radial position r for in-plane torsion of an annular membrane.

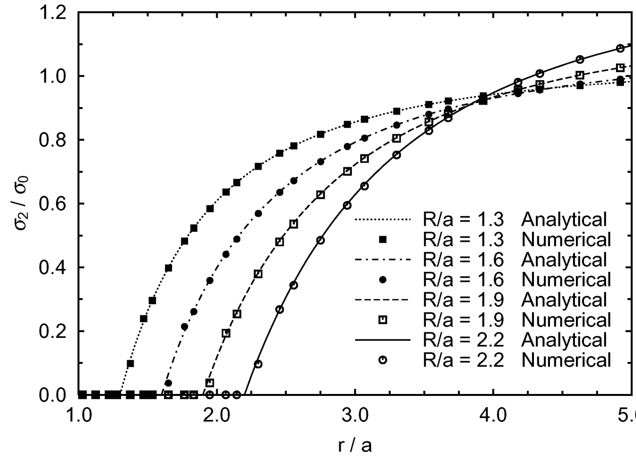

Fig. 12 Minimum principal stress σ_2 vs radial position r for in-plane torsion of an annular membrane.

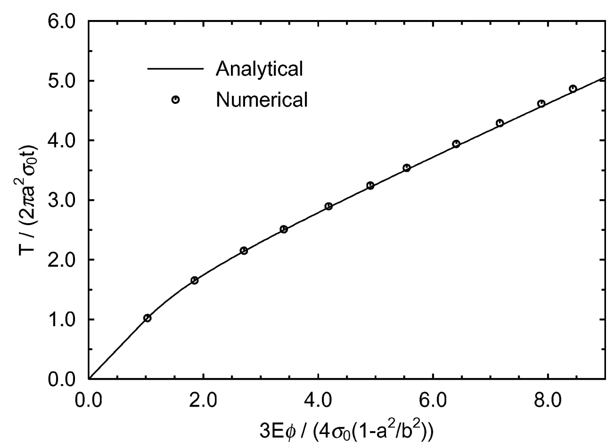

Fig. 13 Relation between applied moment T and angle of twist ϕ for in-plane torsion of an annular membrane.

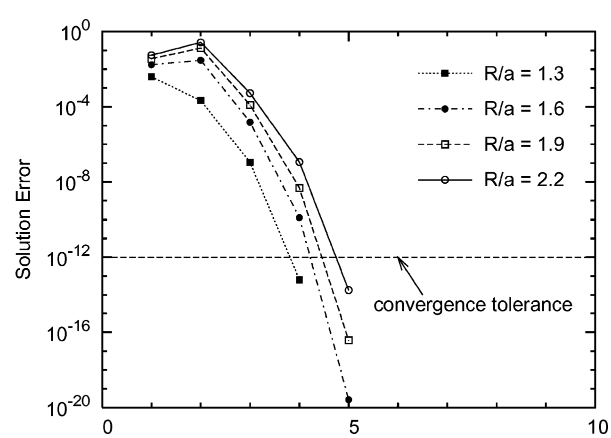

Fig. 14 Convergence behavior for in-plane torsion of an annular membrane.

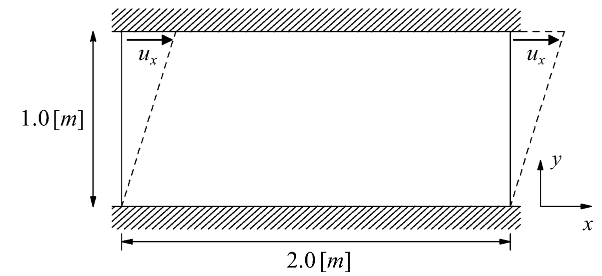

Fig. 15 Rectangular membrane subjected to shear deformation.

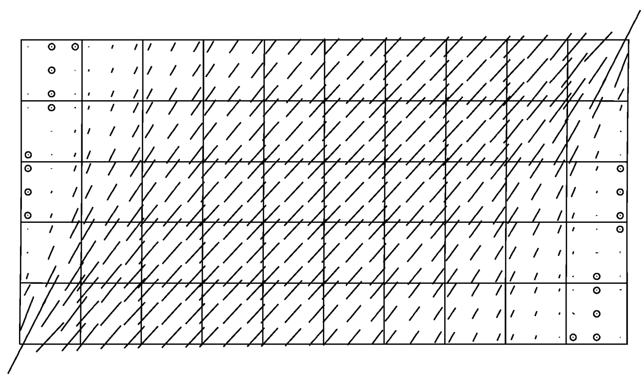

Fig. 16 Principal stresses of the rectangular membrane for displacement $u_x = 5.0$ mm.

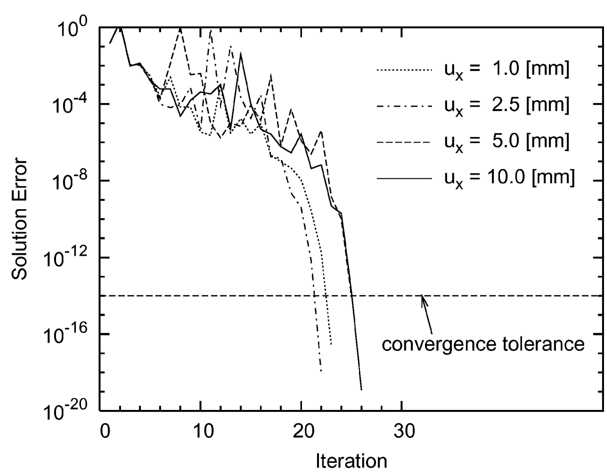

Fig. 17 Convergence behavior for shear deformation of a rectangular membrane.

D. Example 3: Shear Deformation of a Rectangular Membrane

A rectangular membrane, 2.0 m long × 1.0 m wide, is cramped to rigid lower and upper edges with no pretension (Fig. 15). The upper edge is displaced in the x direction by u_x, causing the membrane to be sheared. The thickness, Young's modulus, and Poisson's ratio of the membrane are given as $t = 5.0\ \mu$m, $E = 2.5 \times 10^9$ Pa, and $\nu = 0.3$, respectively. The membrane is discretized using 50 nine-node membrane elements. Finite element calculations using the present treatment of wrinkling are performed for various values of u_x, ranging from 1.0 to 10.0 mm.

Figure 16 shows the principal stresses of the membrane as predicted for $u_x = 5.0$ mm. In the figure, numerical stresses, sampled at Gauss integration points, are represented by vectors that express the magnitude and orientation of the principal stresses. Open circles indicate that the membrane is in slackened state at these points. All of the vectors appear as a single line segment, indicating uniaxial tension state.

Generally, the presence of slackened regions, as depicted in Fig. 16, causes the singularity of the stiffness matrix in the element level. However, the iterative solution process can be successfully performed as long as the assembled global stiffness does not exhibit the singular properties. Figure 17 shows the convergence behavior of the iterative solution processes for this example. In the figure, the solution errors for four different values of u_x are presented. During the iterations, the solution errors tend to fluctuate compared to the earlier two examples. In the last few steps, however, the quadratic convergence is achieved in all four cases.

V. Conclusions

A new modification scheme of the stress–strain tensor for wrinkled membranes is presented based on the tension-field (TF) theory. Modification formulas are given explicitly in matrix-vector form. The proposed modification scheme is straightforwardly integrated into the existing finite element codes that utilize the convected coordinate system, and also, into other finite element codes with a slight modification. The tangent stiffness matrix, obtained from the modified stress–strain tensor, is a consistent one in the sense that it incorporates changes in both the wrinkling direction and the amount of wrinkliness. Numerical analysis using the proposed modification scheme reveals that the scheme provides accurate results while maintaining good convergence properties.

As mentioned, the proposed modification scheme has its basis in the TF theory, where membranes are assumed to have no bending stiffness. Hence, the scheme has the limitation of not being able to predict the fine details of wrinkles themselves (i.e., the amplitude and wavelength of the wrinkles). These fine properties of the wrinkles, though important, are only available through finite element analysis using reliable shell elements, which, on the other hand, usually demands a large amount of computational time. Hence, the proposed scheme, despite the preceding limitation, provides an effective tool for analysis of partly wrinkled membranes.

Though the numerical examples given in this paper are limited to those of two-dimensional problems, the proposed scheme is also applicable to three-dimensional problems in a straightforward manner. Generally speaking, three-dimensional analysis using conventional membrane elements tends to be numerically unstable because of buckling phenomenon. However, we have found that our modification scheme is quite effective to overcome the numerical instabilities encountered in three-dimensional problems. This stabilization effect of the modification scheme will be discussed in a follow-up paper.

Appendix: Calculation of the Components $C^{\alpha\beta\xi\eta}$ (Ref. 21)

Usually, components of the stress–strain tensor $\boldsymbol{C}$ of membranes are expressed in terms of local orthonormal bases, in which case $\boldsymbol{C}$ is decomposed as

$$\boldsymbol{C} = \bar{C}^{\alpha\beta\xi\eta}\bar{\boldsymbol{e}}_\alpha \otimes \bar{\boldsymbol{e}}_\beta \otimes \bar{\boldsymbol{e}}_\xi \otimes \bar{\boldsymbol{e}}_\eta \tag{A1}$$

where $\bar{\boldsymbol{e}}_\alpha$ are orthonormal bases tangent to the membrane surface. If strains in the membrane are assumed to be sufficiently small (this assumption does not exclude large deformations), the components $\bar{C}^{\alpha\beta\xi\eta}$ for isotropic membranes can be expressed in the following familiar form:

$$\begin{bmatrix} \bar{C}^{1111} & \bar{C}^{1122} & \bar{C}^{1112} \\ \bar{C}^{2211} & \bar{C}^{2222} & \bar{C}^{2212} \\ \bar{C}^{1211} & \bar{C}^{1222} & \bar{C}^{1212} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{A2}$$

Other components not appearing in Eq. (A2) follow from the symmetric property of $\bar{C}^{\alpha\beta\xi\eta}$. The components $C^{\alpha\beta\xi\eta}$ in Eq. (10) can be calculated from $\bar{C}^{\alpha\beta\xi\eta}$ by a transformation of coordinates as

$$C^{\alpha\beta\xi\eta} = \bar{C}^{\lambda\pi\mu\nu}\left(\bar{\boldsymbol{e}}_\lambda \cdot \boldsymbol{G}^\alpha\right)\left(\bar{\boldsymbol{e}}_\pi \cdot \boldsymbol{G}^\beta\right)\left(\bar{\boldsymbol{e}}_\mu \cdot \boldsymbol{G}^\xi\right)\left(\bar{\boldsymbol{e}}_\nu \cdot \boldsymbol{G}^\eta\right) \tag{A3}$$

Acknowledgments

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S. Saigal
Associate Editor